Advanced Microeconomics

Fall 2024

Midterm Exam

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1. You have a total of 80 minutes (1 hour and 20 minutes) to solve the exam.
2. The use of calculators is not allowed.
3. If you need additional space to answer a question, you can use the back *of the same page*.

Read each question carefully. Good luck!

I (4.5 points)

Consider an economy with two agents, A and B, with preferences over two goods as follows:

$$U_A = x_A^{\frac{1}{3}} \cdot y_A^{\frac{2}{3}}$$

$$U_B = \min\{2x_B, 3y_B\}$$

It is known that the initial endowment allocation is $(\omega_x^A, \omega_y^A, \omega_x^B, \omega_y^B) = (15, 0, 5, 15)$.

a) (2 points) Find the walrasian equilibrium in this economy.
Solution:
Agent A:

$$\max_{\{x_A, y_A\}} U_A = x_A^{\frac{1}{3}} \cdot y_A^{\frac{2}{3}} \text{ s.t. } x_A P_x + y_A P_y = 15P_x + 0P_y$$

Share Rule:

$$\begin{cases} x_A^* = \frac{1}{3} \cdot \frac{15P_x + 0P_y}{P_x} \\ y_A^* = \frac{2}{3} \cdot \frac{15P_x + 0P_y}{P_y} \end{cases} \Leftrightarrow \begin{cases} x_A = 5 \\ y_A = \frac{10P_x}{P_y} \end{cases}$$

Agent B:

$$\max_{\{x_B, y_B\}} U_B = \min\{2x_B, 3y_B\} \text{ s.t. } x_B P_x + y_B P_y = 5P_x + 15P_y$$

$$\begin{cases} 2x_B = 3y_B \\ x_B P_x + y_B P_y = 5P_x + 15P_y \end{cases} \Leftrightarrow \begin{cases} y_B = \frac{2x_B}{3} \\ x_B P_x + \frac{2x_B}{3} P_y = 5P_x + 15P_y \end{cases}$$

$$\begin{cases} y_B = \frac{2x_B}{3} \\ x_B(P_x + \frac{2P_y}{3}) = 5P_x + 15P_y \end{cases} \Leftrightarrow \begin{cases} y_B = \frac{2}{3} \cdot \frac{5P_x + 15P_y}{P_x + \frac{2P_y}{3}} = \frac{10P_x + 30P_y}{3P_x + 2P_y} \\ x_B = \frac{5P_x + 15P_y}{P_x + \frac{2P_y}{3}} = \frac{15P_x + 45P_y}{3P_x + 2P_y} \end{cases}$$

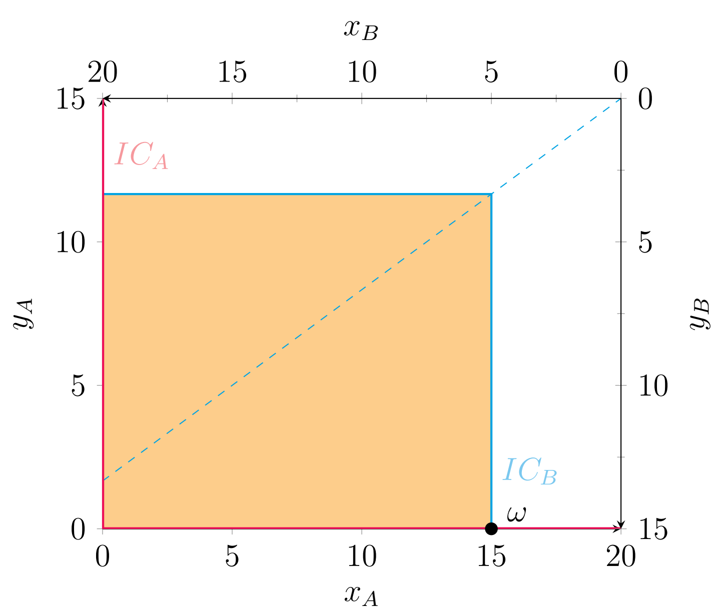
Equilibrium:

$$x_A^* + x_B^* = 20 \Leftrightarrow 5 + \frac{15P_x + 45P_y}{3P_x + 2P_y} = 20 \Leftrightarrow 15P_x + 45P_y = 45P_x + 30P_y \Leftrightarrow$$
$$\Leftrightarrow 30P_x = 15P_y \Leftrightarrow \frac{P_x}{P_y} = \frac{1}{2} \rightarrow \text{ Equilibrium Price Ratio}$$

$$x_A^* = 5; \quad x_B^* = 20 - 5 = 15; \quad y_A^* = \frac{10 \cdot 1}{2} = 5; \quad y_B^* = 15 - 5 = 10$$
$$\left(x_A^*, y_A^*, x_B^*, y_B^*, \frac{1}{2}\right) = \left(5, 5, 15, 10, \frac{1}{2}\right)$$

Grading: 0.75 points for computing the demand functions of Agent A; 0.75 point for computing the demand functions of Agent B; 0.5 points for computing the equilibrium allocation and price ratio.

b) (1 point) Find the Mutual Advantages Set graphically.
<u>Solution:</u>



Grading: 0.25 for the initial endowment, 0.25 for Agent A's indifference curve, 0.25 for Agent B's indifference curve, 0.25 for shading the MAS.

c) (1 points) Without additional calculations, can you state whether the location found in a) is Pareto Efficient?

- Both agents have weakly monotonic preferences, thus we can apply the First Welfare Theorem
- The allocation found in exercise a) is a walrasian equilibrium

- Under the conditions of the First Welfare Theorem, any walrasian equilibrium found in an economy will be a Pareto Efficient allocation

Grading: 0.5 for the conditions for the First Welfare Theorem; 0.5 for applying the First Welfare Theorem

II (5 points)

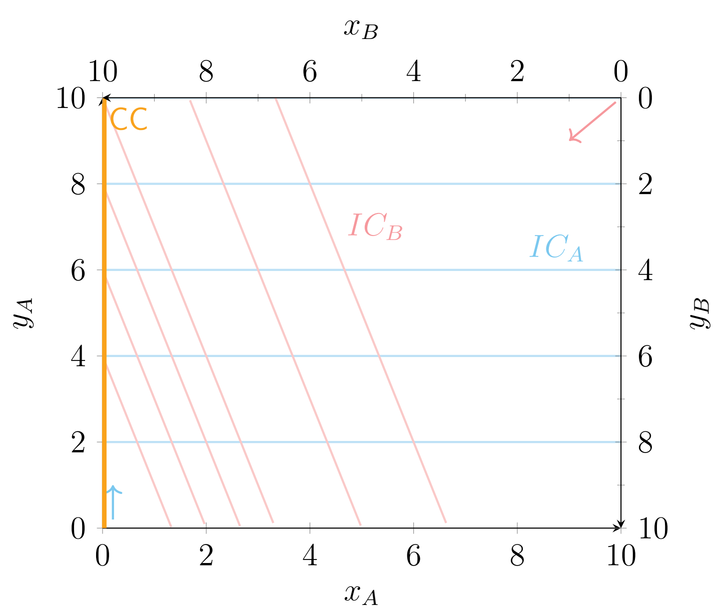
Consider an economy with two agents, A and B, with preferences over two goods as follows:

$$U_A = y_A$$

$$U_B = 3x_B + y_B$$

The total endowments of x and y are given by $(\omega_x, \omega_y) = (10, 10)$.

a) (2 points) Represent the contract curve graphically.
<u>Solution:</u>



CC: $x_A = 0$ (or $x_B = 10$)

Grading: 1 point for the justification; 1 point for the graphical representation of the contract curve

b) (1.5 points) Show that the UPF is given by the function $U_A = \begin{cases} 40 - U_B & , U_B \geq 30 \\ 10 & , U_B < 30 \end{cases}$.
(Hint: remember to consider the maaximum utility that agent A can achieve in this economy.)
<u>Solution:</u>

$$x_A = 0 \implies x_B = 10 \to U_B = 3 \times 10 + y_B$$
$$y_A + y_B = 10 \Leftrightarrow y_B = 10 - y_A \to U_B = 30 + (10 - y_A) = 40 - y_A$$
$$U_A = y_A \to U_B = 40 - U_A$$

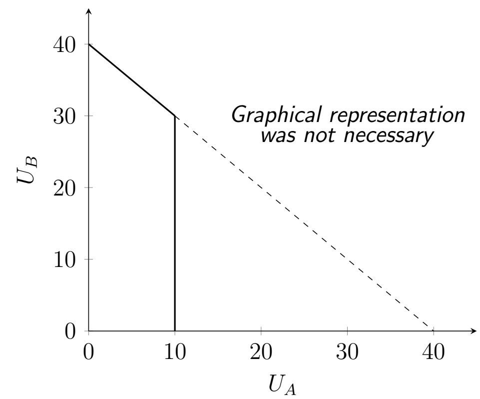
However:

$$\omega_y = 10 \wedge U_A = y_A \implies U_A \in [0, 10]$$

meaning there is a limit to A's utility which is not yet being considered by $U_B = 40 - U_A$ Since the maximum utility B can achieve when $U_A = 10$ is given by $10 = 40 - U_B \Leftrightarrow U_B = 30$, we know there must be a segment on the UPF where $U_A = 10 \wedge U_B \in [0, 30]$, thus:

$$U_B = 40 - U_A \Leftrightarrow U_A = 40 - U_B$$

$$U_A = \begin{cases} 40 - U_B & , U_B \geq 30 \\ 10 & , U_B < 30 \end{cases}$$



Grading: 0.5 for using the CC; 0.5 for using the feasibility conditions; 0.5 justifying the ceiling for A

c) (1.5 points) After drawing the UPF, identify graphically the choice that would be made by a Cobb-Douglas social welfare function, where welfare $W = U_A \cdot U_B$. Could the associated allocation be achieved as a walrasian equilibrium?

Solution:

$$\max_{\{U_A, U_B\}} W = U_A \cdot U_B \text{ s.t. } U_A = \begin{cases} 40 - U_B & , U_B \geq 30 \ (1) \\ 10 & , U_B < 30 \ (2) \end{cases}$$

Attempting to maximize in case 1, where $U_B \geq 30$:

$$W = (40 - U_B) \cdot U_B = 40U_B - U_B^2$$

$$\text{F.O.C.: } \frac{\delta W}{\delta U_B} = 0 \Leftrightarrow 40 - 2U_B = 0 \Leftrightarrow U_B = 20 \implies U_A = 40 - 20 = 20$$

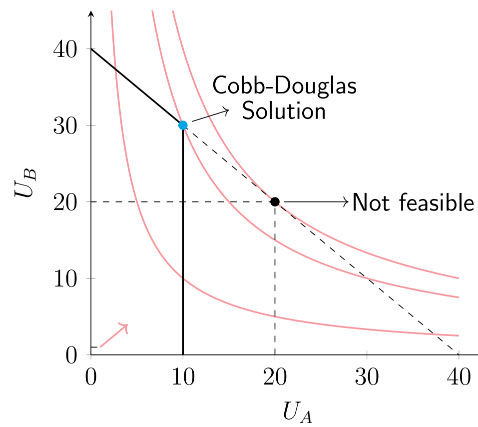
However $U_B < 30$ and $U_A > 10 \to$ Not feasible

Attempting to maximize in case 2, where $U_B < 30$:

$$W = 10 \cdot U_B$$

$$\text{F.O.C.: } \frac{\delta W}{\delta U_B} = 10 \to \text{Strictly increasing in } U_B$$

Set U_B as high as possible: $U_B = 30 \wedge U_A = 10$



Note: Only the graphical solution was required

The solution is found on the portion of the UPF associated with the contract curve, thus we know the associated allocation will be efficient (could also find the respective allocation to be $(x_A, y_A, x_B, y_B) = (0, 10, 10, 0)$ and demonstrate it was part of the contract curve).

As both agents have weakly monotonic and weakly convex preferences, the Second Welfare Theorem applies, thus ensuring that any Pareto Efficient allocation can become a walrasian equilibrium through a given redistribution of the initial endowments.

Grading: 0.5 points for the graphical solution; 0.5 points conditions of the Second Welfare Theorem; 0.5 points for applying the Second Welfare Theorem

III (3.5 points)

Vasco is a very dedicated student who enjoys keeping his studying notes very organized, both because he enjoys how appealing they look and because it helps him when reviewing for the evaluations. The more effort he puts into his notes, the more accurate they are and better organized they are, although more effort means spending more time and energy when drafting him, thus earning him a total benefit of $12\sqrt{e}$ and a total cost of $3e$.

Additionally, Vasco has 2 friends with whom he shares his notes. As this helps them review the contents of the classes as well, each of them **individually** benefits from his effort in a value of $6\sqrt{e}$

a) (1 point) What level of effort will Vasco choose to employ?
Solution:
Private Benefit: $12\sqrt{e}$; Private Cost: $3e$; External Benefit: $2 \times 6\sqrt{e}$
Vasco solves:

$$MB^{private} = MC^{private}$$
$$\frac{6}{\sqrt{e}} = 3$$
$$e = 4$$

As such, he will choose $e = 4$.

Grading:0.5 for the MB=MC condition; 0.5 for the conclusion

b) (1.5 points) Find the socially optimal level of effort. Compare and interpret the Private and Social solutions. Are they different? Why?
Solution:

$$\text{Samuelson Condition: } \frac{6}{\sqrt{e}} + \frac{3}{\sqrt{e}} + \frac{3}{\sqrt{e}} = 3$$
$$e^* = 16$$

Or:

$$MB^{external} = 2 \times \frac{1}{2} \times \frac{6}{\sqrt{e}} = \frac{6}{\sqrt{e}}$$
$$MB^{social} = \frac{6}{\sqrt{e}} + \frac{6}{\sqrt{e}} = \frac{12}{\sqrt{e}} \text{ and } MC^{social} = 3$$

At the socially optimal level of effort:

$$MB^{social} = MC^{social}$$
$$\frac{12}{\sqrt{e}} = 3$$
$$e = 16$$

As we can see, the socially optimal level of effort is substantially greater than the privately optimal amount. This occurs as Vasco is not currently internalizing the positive externality produced by his effort.

Grading: 1 for the right MB and MC; 0.5 for the conclusion

c) (1 point) How could you lead Vasco to put the socially optimal level of effort?
Solution:

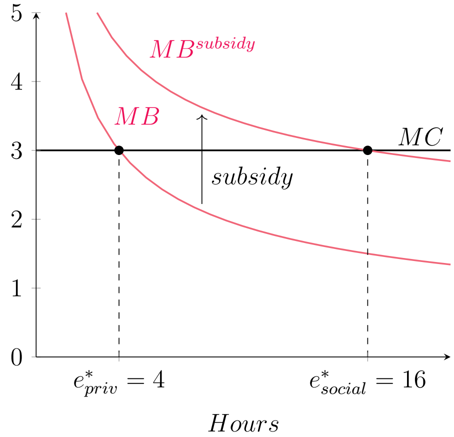
Method 1: Negotiation, applying the Coase Theorem (the 2 friends could pay Vasco for additional effort)

Method 2:
We could impose a Pigouvian Subsidy: a unit subsidy on the level of effort chosen by Vasco. Find the External Marginal Benefit at the socially optimal level of effort:

$$MB^{external} = \frac{6}{\sqrt{e}}|_{(e^*=16)} = \frac{6}{\sqrt{16}} = \frac{3}{2}$$

Set the unit subs equal to the Marginal Benefit at the socially optimal amount:

$$s^* = \frac{3}{2}$$



Method 3: Lindhal Subsidy: Charge each agent the monetary value of their net benefit/cost at the socially optimal level:

$$\text{Vasco: } \frac{6}{\sqrt{16}} - 3 = -\frac{6}{4}$$
$$\text{Each friend: } \frac{3}{\sqrt{16}} = \frac{3}{4}$$

Each friend will pay Vasco a unit subsidy of $\frac{3}{4}$, compensating the net marginal cost of $\frac{6}{4}$

Grading: 0.5 for a possible solution; 0.5 for the complete description

V (7.5 points)

Consider the following two-player simultaneous game:

1 \ 2	C	D
C	3,3	-1,1
D	1,-1	0,0

a) (2.5 points) Compute <u>all</u> the Nash Equilibria of the simultaneous game.

	1 \ 2	C (q)	D ($1-q$)
p	C	3,3	-1,1
$1-p$	D	1,-1	0,0

<u>For Agent 1:</u>

$$E_\pi(C) = 3 \times q - 1 \times (1-q) = 4q - 1$$
$$E_\pi(D) = 1 \times q + 0 \times (1-q) = q$$

Play C if $4q - 1 > q \Leftrightarrow q > \frac{1}{3}$, or D if $q < \frac{1}{3}$.

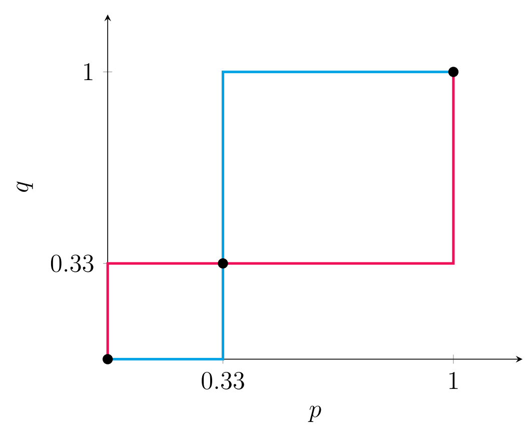
Thus 1's Best Response is to set $p = \begin{cases} 0 & , q < \frac{1}{3} \\ [0,1] & , q = \frac{1}{3} \\ 1 & , q > \frac{1}{3} \end{cases}$

For Agent 2:

$$E_\pi(C) = 3 \times p - 1 \times (1-p) = 4p - 1$$
$$E_\pi(D) = 1 \times p + 0 \times (1-p) = p$$

Play C if $4p - 1 > p \Leftrightarrow p > \frac{1}{3}$, or D if $p < \frac{1}{3}$.

Thus 2's Best Response is to set $q = \begin{cases} 0 & , p < \frac{1}{3} \\ [0,1] & , p = \frac{1}{3} \\ 1 & , p > \frac{1}{3} \end{cases}$



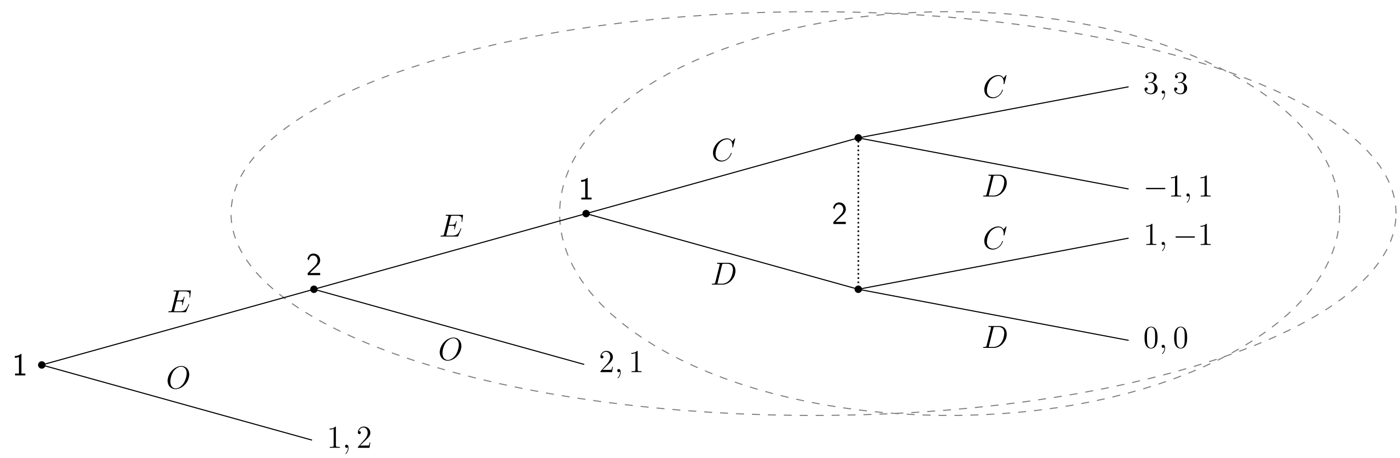
Nash equilibria are characterized by $(p = 1, q = 1)$, $(p = 1/3, q = 1/3)$, and $(p = 0, q = 0)$.

Grading: 0.75 for the best responses of 1; 0.75 for the best responses of 2; 1 for the correct conclusion. (1 point in case only pure-strategy equilibria are calculated)

Assume now that this game is only played at time $t = 3$, after two sequential moves.

At time $t = 1$, player 1 chooses to enter (E) or to stay out (O). If player 1 chooses to stay out, the game ends and she gets a payoff of **1**, and player 2 gets a payoff of **2**.
At time $t = 2$, in case player 1 chose E at $t = 1$, player 2 observes player's 1 action and decides whether to enter (E) or to stay out (O). If player 2 chooses to stay out, the game ends and the payoffs are **2** for player 1, and **1** for player 2.
At time $t = 3$, if player 1 chose E at t=1 and player 2 also chose E at $t = 2$, then both players play the simultaneous game you solved in a.

b) (2 points) Represent this sequential game in extensive form. How many subgames are there?
Solution: In total, there are 3 subgames:



1. The whole game
2. The subgame after Player 1 plays E
3. The subgame after Player 1 plays E and 2 plays E

Grading: 1.25 for the game tree; 0.75 for correct identification of all subgames

c) (3 points) Find all the SPNE for this game.
Solution:

We know the NE of the subgame that begins after 1 plays E and 2 plays E.

If the NE is $(p = 1, q = 1)$, or (C,C), payoffs would be (3,3), and therefore 2 chooses E in the previous subgame and 1 chooses E at the beginning. We have an SPNE: (EC, EC)

If the NE is $(p = 0, q = 0)$, or (D,D), payoffs would be (0,0), and therefore 2 chooses O in the previous subgame and 1 chooses E at the beginning. We have an SPNE: (ED, OD)

If the NE is $(p = \frac{1}{3}, q = \frac{1}{3})$, payoffs would be :
For Player 1: $E_\pi(C) = E_\pi(D) = \frac{1}{3}$
For Player 2: $E_\pi(C) = E_\pi(D) = \frac{1}{3}$

$(\frac{1}{3}, \frac{1}{3})$, and therefore 2 chooses O in the previous subgame and 1 chooses E at the beginning. We have an SPNE: $((E, p = \frac{1}{3}), (O, q = \frac{1}{3}))$

Grading: 1 for each SPNE