Group I (4.5 points)

Consider an economy with two agents, A and B, with preferences over two goods as follows:

$$U_A = 2\ln(x_A) + \ln(y_A)$$

$$U_B = \min\{x_B, y_B\}$$

Agent A initially owns 3 units of x and 3 units of y. Agent B initially owns 7 units of x and 7 units of y.

a) (2 points) Find the walrasian equilibrium in this economy.
Solution:
Agent A:

$$\max_{\{x_A, y_A\}} U_A = 2\ln(x_A) + \ln(y_A) \text{ s.t. } 3P_x + 3P_y = x_A P_x + y_A P_y$$

$$\begin{cases} MRS^A_{x,y} = \frac{P_x}{P_y} \\ 3P_x + 3P_y = x_A P_x + y_A P_y \end{cases} \Leftrightarrow \begin{cases} x_A = 2 + 2\frac{P_y}{P_x} \\ y_A = 1 + \frac{P_x}{P_y} \end{cases}$$

Agent B:

$$\max_{\{x_B, y_B\}} U_B = \min\{x_B, y_B\} \text{ s.t. } 7P_x + 7P_y = x_B P_x + y_B P_y$$

$$\begin{cases} x_B = y_B \\ 7P_x + 7P_y = x_B P_x + y_B P_y \end{cases} \Leftrightarrow \begin{cases} x_A = 7 \\ y_A = 7 \end{cases}$$

Equilibrium:

$$y_A + y_B = 10 \Leftrightarrow 1 + \frac{P_x}{P_y} + 7 = 10 \Leftrightarrow \frac{P_x}{P_y} = 2$$

$$(x_A^*, y_A^*, x_B^*, y_B^*) = (3, 3, 7, 7)$$

Grading: 0.75 points for computing the demand functions of Agent A; 0.75 point for computing the demand functions of Agent B; 0.5 points for computing the equilibrium allocation and price ratio.

b) (1 point) Show that the core is composed of a single allocation.
Solution:

The core is the intersection of the contract curve with the mutual advantages region. The endowment point coincides with the mutual advantages region. Since it is also on the contract curve, it coincides with the core.

Grading: 0.25 for definition of core, 0.25 for the mutual advantages, 0.5 for stating the endowment is efficient.

c) (1.25 points) Would it be possible to implement the allocation $(x_A, y_A, x_B, y_B) = (5, 5, 5, 5)$ as a walsarian equilibrium if the initial endowment were redistributed? If so, what would be the equilibrium price ratio?

- The allocation $(x_A, y_A, x_B, y_B) = (5, 5, 5, 5)$ belongs to the Contract Curve $x_A = y_A$ and thus is Pareto Efficient.
- According to the 2nd Welfare Theorem, an efficient allocation can be reached as a Walrasian Equilibrium through redistribution of the initial endowment. We can apply the 2nd Welfare Theorem, as preferences are weakly monotonic and convex.
- The equilibrium price ratio would be $\frac{P_x}{P_y} = 2$.

Grading: 0.3 points for justifying that the allocation is PE; 0.3 points for referring to the 2nd Welfare Theorem; 0.3 points for concluding that it is possible to apply the 2nd Welfare Theorem (as preferences are weakly monotonic and convex); 0.35 points for the price ratio

Group II (4.75 points)

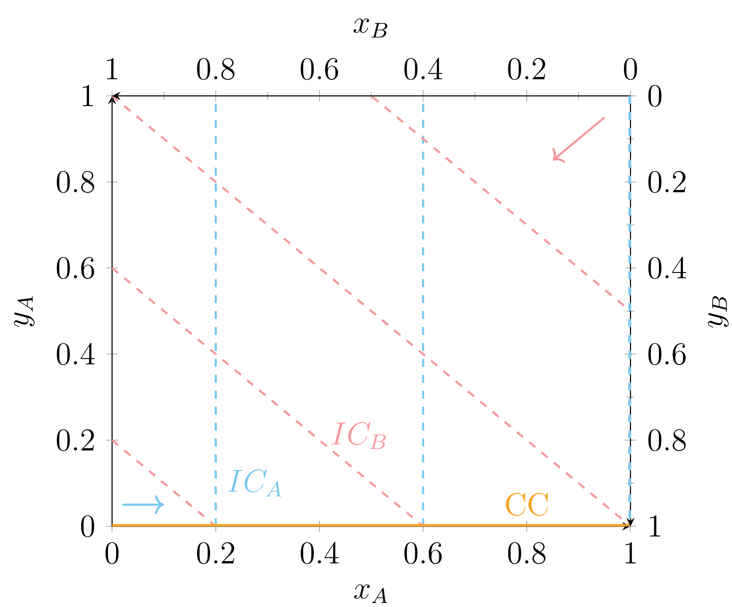
Consider an economy with two agents, A and B, with preferences over two goods as follows:

$$U_A = x_A^2$$

$$U_B = x_B + y_B$$

It is known the initial endoment is $\omega_x^A + \omega_x^B = 1$ and $\omega_y^A + \omega_y^B = 1$.

a) (1.75 points) Find the contract curve and represent it graphically in the Edgeworth Box. **Solution:**



$$\text{CC: } y_A = 0 \vee y_B = 1$$

Grading: 0.625 points for the justification; 0.625 points for the graphical representation of the contract curve; 0.5 points for the analytical expression.

b) (1.5 points) Find and draw the utility possibilities frontier.
Solution:

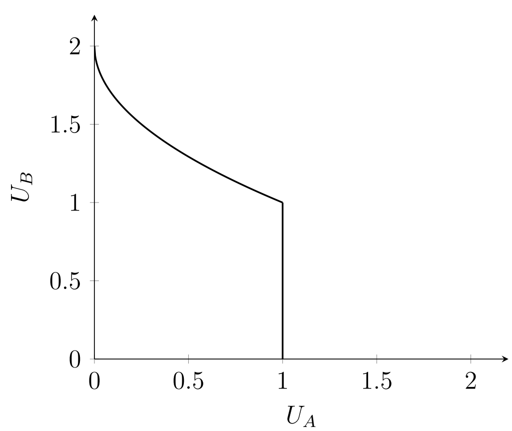
$$U_A = x^2 \Leftrightarrow x_A = \sqrt{U_A}, 0 \leq U_A \leq 1$$

$$\begin{aligned} \text{CC: } y_B = 1 \Rightarrow U_B &= x_B + 1 \\ &= (1 - x_A) + 1 \\ &= 2 - \sqrt{U_A} \end{aligned}$$

$$U_B = 2 - \sqrt{U_A}, 0 \leq U_A \leq 1$$

Note how the contract curve does not consider the allocations where B's utility is below 1 (only provides the curved line on the graph below), and thus the UPF must be completed by considering that, for a level of $U_B \leq 1$, $U_A = 1$ (consider above the indifference curves where $U_B \leq 1$ and look for the greatest level of utility A can achieve):

$$U_A = \begin{cases} 1 & , 0 \leq U_B < 1 \\ (2 - U_B)^2 & , 1 \leq U_B \leq 2 \end{cases}$$



Grading: 0.3 for using the CC; 0.3 for using the feasibility conditions; 0.3 for writing an expression relating agents' utilities; 0.3 for recognizing that $0 \leq U_A \leq 1$; 0.3 for drawing the UPF.

c) (1.5 points) Find a socially optimal allocation under a rawlsian social welfare function. (If you haven't replied to b), assume that the UPF is given by $U_B = 2 - \sqrt{U_A}$)
Solution:

$$\max_{\{U_A, U_B\}} W = \min\{U_A, U_B\} \text{ s.t. } U_B = 2 - \sqrt{U_A}$$

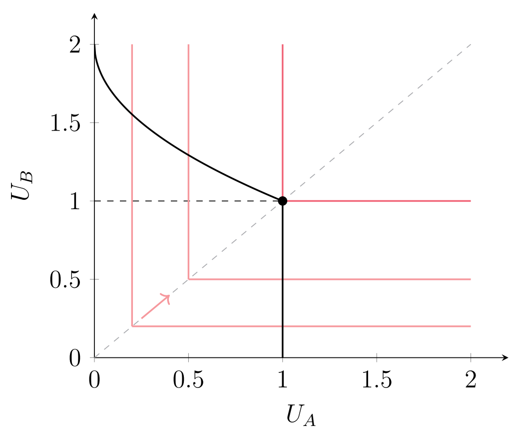
$$\Rightarrow \begin{cases} U_A = U_B \\ U_B = 2 - \sqrt{U_A} \end{cases} \quad \Rightarrow U_A = 2 - \sqrt{U_A} \Leftrightarrow$$

$$U_A + \sqrt{U_A} - 2 = 0 \Leftrightarrow \sqrt{U_A} = \frac{-1 \pm \sqrt{1 - 4 \times (-2)}}{2}$$
$$\sqrt{U_A} = \frac{-1-3}{2} \vee \sqrt{U_A} = \frac{-1+3}{2}$$
$$\sqrt{U_A} = -2 \vee \sqrt{U_A} = 1$$

$$U_A = 1 \Rightarrow U_B = 2 - 1 = 1$$
$$\Rightarrow x_A^* = 1 \Rightarrow x_B^* = 0$$

$$U_B = 1 \wedge x_B^* = 0 \Rightarrow 0 + y_B^* = 1$$
$$y_B^* = 1 \Rightarrow y_A^* = 0$$

$$(x_A^*, y_A^*, x_B^*, y_B^*) = (1, 0, 0, 1)$$



Grading: 0.5 points for the formalization of the problem; 0.5 points for the optimal utilities; 0.5 points for reaching an optimal allocation.

Group III (3.75 points)

The owners of Bairro Alto must decide its opening hours. Assume that for a total of h hours, their revenue will be $16h$, while the cost they face is given by h^2.
However, patrons tend to be very loud on the street - and the music from the bar will also be loud. This causes a cost of h^2 for those who live in the neighborhood.

a) (1 point) For how many hours will the owners decide to keep the bar open?
Solution:
Private Benefit: $16h$; Private Cost: h^2; External Cost: h^2
At the private optimal point:

$$MB^{private} = MC^{private}$$
$$16 = 2h$$
$$h = 8$$

As such, they will desire to keep the bar open for 8 hours.

Grading: 0.25 for the private marginal benefit; 0.25 for the private marginal cost; 0.25 for the utility maximizing condition; 0.25 for the conclusion.

b) (1.5 points) Find the socially optimal number of hours. Compare and interpret the Private and Social solutions. Are they different, and, if so, why?
Solution:

$$MB^{social} = 16h \text{ and } MC^{social} = 2h + 2h = 4h$$

At the socially optimal number of hours:

$$MB^{social} = MC^{social}$$
$$16 = 4h$$
$$h = 4$$

As we can see, the socially optimal number of hours is substantially lower than the privately optimal amount (half of the latter, to be precise). This occurs as the owners of the bar are not internalizing the negative externality present, and thus keep it open for a greater number of hours than the socially optimal.

Grading: 0.3 for the social marginal benefit; 0.5 for the social marginal cost; 0.3 for the utility maximizing condition; 0.4 for the conclusion.

c) (1.25 points) If you were tasked with designing a taxing system that would lead to the efficient solution, what would you recommend?
Solution: We could impose a Pigouvian Tax: a unit tax on the number of hours the bars are open.
Method 1:

$$MC_t^{private} = 2h + t$$

$$\begin{cases} MB^{private} = MC_t^{private} \\ h^* = 4 \end{cases} \Leftrightarrow \begin{cases} 16 = 2h + t \\ h^* = 4 \end{cases}$$

$$\begin{cases} t^* = 8 \\ h^* = 4 \end{cases}$$

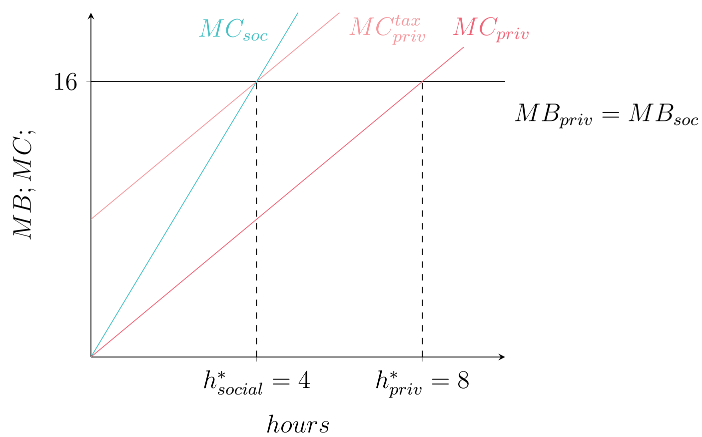
Grading: 0.5 or identifying the Pigouvian Tax; 0.5 for formalizing the system of equations; 0.25 for the conclusion.

Method 2:
Find the External Marginal Cost at the socially optimal number of hours:

$$MC^{external} = 2h|_{(h^*=4)} = 2 \times 4 = 8$$

Set the unit tax equal to the Marginal Cost at the socially optimal amount:

$$t^* = 8$$



Grading: 0.5 points for identifying the Pigouvian Tax; 0.5 for finding the external marginal cost at the social optimum; 0.25 for the conclusion.

Group IV (2.5 points)

Consider a market with two firms, where both choose quantities simultaneously. Demand in the market is given by $P = 5 - Q$. Firm 1 has a marginal cost of 1 (and no fixed costs). Firm 2 has a marginal cost of 0 (and no fixed costs).
Find the Nash Equilibrium of this game.

Solution:

Cournot Game
Firm 1's Best Response function:

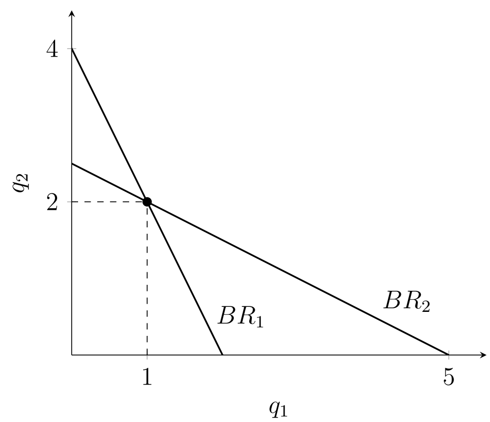
$$max_{\,q1}\, \pi_1 = [5 - (q_1 + \overline{q_2})]q_1 - q_1$$
$$FOC : 5 - 2q_1 - \overline{q_2} - 1 = 0$$
$$q_1^* = \frac{4 - \overline{q_2}}{2}$$

Firm 2's Best Response function:

$$max_{\,q2}\,\pi_2 = [5 - (\overline{q1} + q_2)]q_2 - 0 \times q_2$$
$$FOC : 5 - \overline{q_1} - 2q_2 = 0$$
$$q_2^* = \frac{5 - \overline{q_1}}{2}$$

To find the equilibrium:

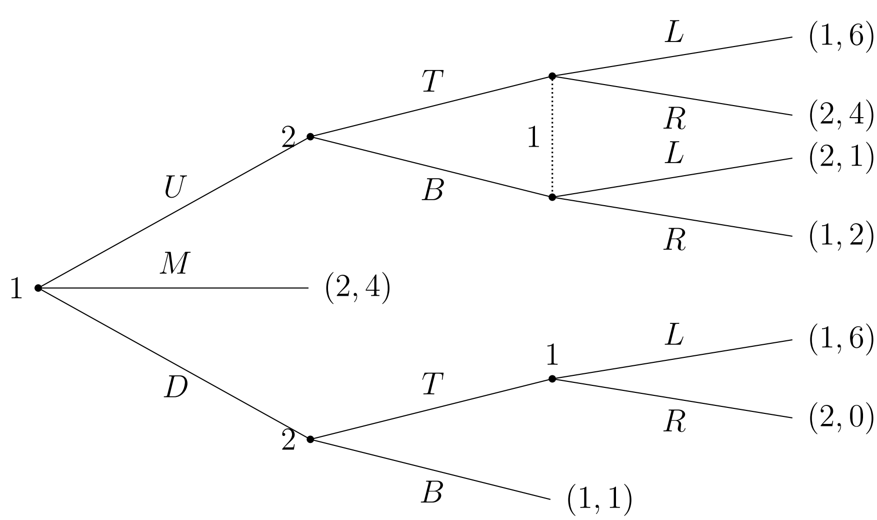
$$\begin{cases} q_1^* = \frac{4-\overline{q_2}}{2} \\ q_2^* = \frac{5-\overline{q_1}}{2} \end{cases} \Leftrightarrow \begin{cases} q_1^* = 1 \\ q_2^* = 2 \end{cases}$$



Grading: 0.75 for finding Firm 1's Best Response; 0.75 for finding Firm 2's Best Response; 1 for calculating the equilibrium.

Group V (4.5 points)

Consider the following game in extensive form:

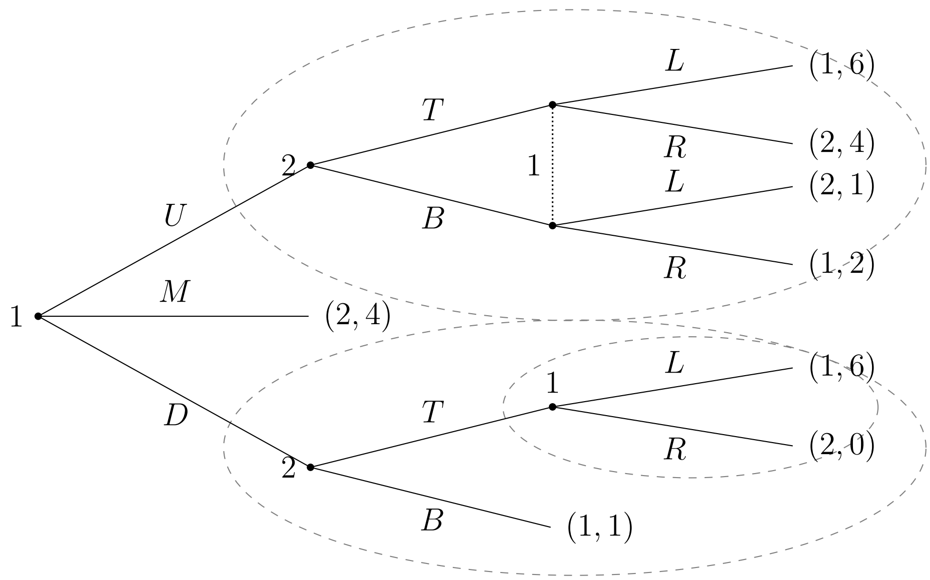


a) (0.75 points)How many pure strategies does player 1 have in this game?
Solution: Player 1 has $3 \times 2 \times 2 = 12$ pure strategies.

Grading: 0.75 for calculating the number of pure strategies.

b) (0.75 points) Identify all the subgames in this game.
Solution: In total, there are 4 subgames:

1. The whole game
2. The subgame after Player 1 plays U
3. The subgame after Player 1 plays D
4. The subgame after Player 1 Plays D and Player 2 plays T



Grading: 0.25 for partial identification of correct subgames; 0.75 for correct identification of all subgames

c) (3 points) Find all the SPNE for this game.
Solution: Let us begin by solving the Subgame after Player 1 plays U:

1 \ 2	T	B
L	1,<u>6</u>	<u>2</u>,1
R	**<u>2</u>,<u>4</u>**	1,2

The Nash Equilibrium in Pure Strategies in this game is (R, T). Furthermore, T is a strictly dominating strategy for Player 2, and therefore the unique Nash Equilibrium is (R, T)

Next, the solution for the subgame after Player 1 plays D and Player 2 plays T is for Player 1 to play R.

Thus, the solution for the subgame after Player 1 plays D is for Player 2 to play B.

And finally, Player 1 has two Best Responses regarding his initial decision: to play U and to play M, thus yielding two SPNE: (URR, TB) and (MRR, TB).

The SPNE are therefore (URR, TB), (MRR, TB) and $[(p, R, R), (TB)], p \in [0, 1]$, where U is played with probablity p and M is played with probability $1 - p$.

Grading: 0.25 for the subgame after (D, T); 0.25 for the subgame after D; 1 for the subgame after U (including matrix and argument); 0.5 for the conclusion about 1's move at the beginning of the game; 1 for correctly identifying the two SPNE