games
with incomplete information

static games: Bayes-Nash equilibrium

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- Harsanyi’s proposal
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dynamic games: Perfect Bayesian Equilibrium

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- example of a dynamic game with incomplete information

example:
prisoner's dilemma – 2 is tough

1 \ 2 type 1	Not to confess	Confess
Not to confess	-1,-1	-6,0
Confess	0,-6	-3,-3

example:
prisoner's dilemma – 2 is accommodating

1 \ 2 (type 2)	Not to confess	Confess
Not to confess	-1,1	-6,-2
Confess	0,-4	-3,-5

static games with incomplete information: Bayes-Nash equilibrium

A game of incomplete information is one in which players do not know some relevant characteristic of their opponents, which may include their payoffs, their available options, and even their beliefs

Solution?

Ex: Prisoner's dilemma

Player 1 always plays Confess

Player 2, type tough plays Confess

Player 2, type accommodating plays Not Confess

example:
coordination game: 2 is matched

1 \ 2 type 1	Book launch	Movie
Book launch	2,1	0,0
Movie	0,0	1,2

example:
coordination game: 2 is mismatched

1 \ 2 (type 2)	Book launch	Movie
Book launch	2,0	0,1
Movie	0,2	1,0

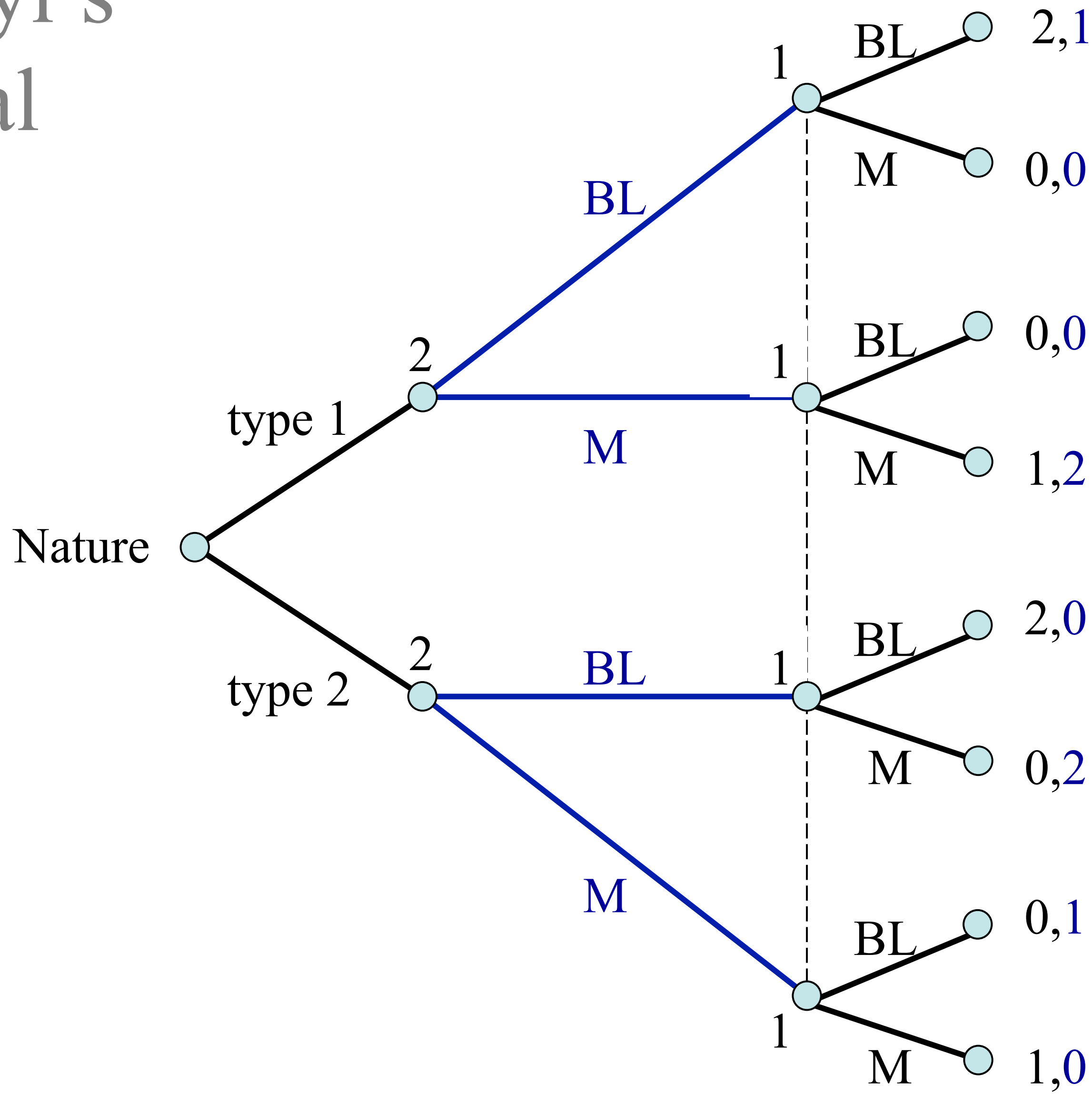
static games with incomplete information: Bayes-Nash equilibrium

Ex: Coordination game

Can player 1 play BL in a Bayes-Nash equilibrium?
Player 2, type matched will play BL (best response)
Player 2, type mismatched plays M (best response)

Assumption of a common prior:
probabilities of types must become part of the game and are known by all players

Harsanyi's proposal



static games with incomplete information: Bayes-Nash equilibrium

Harsanyi's proposal

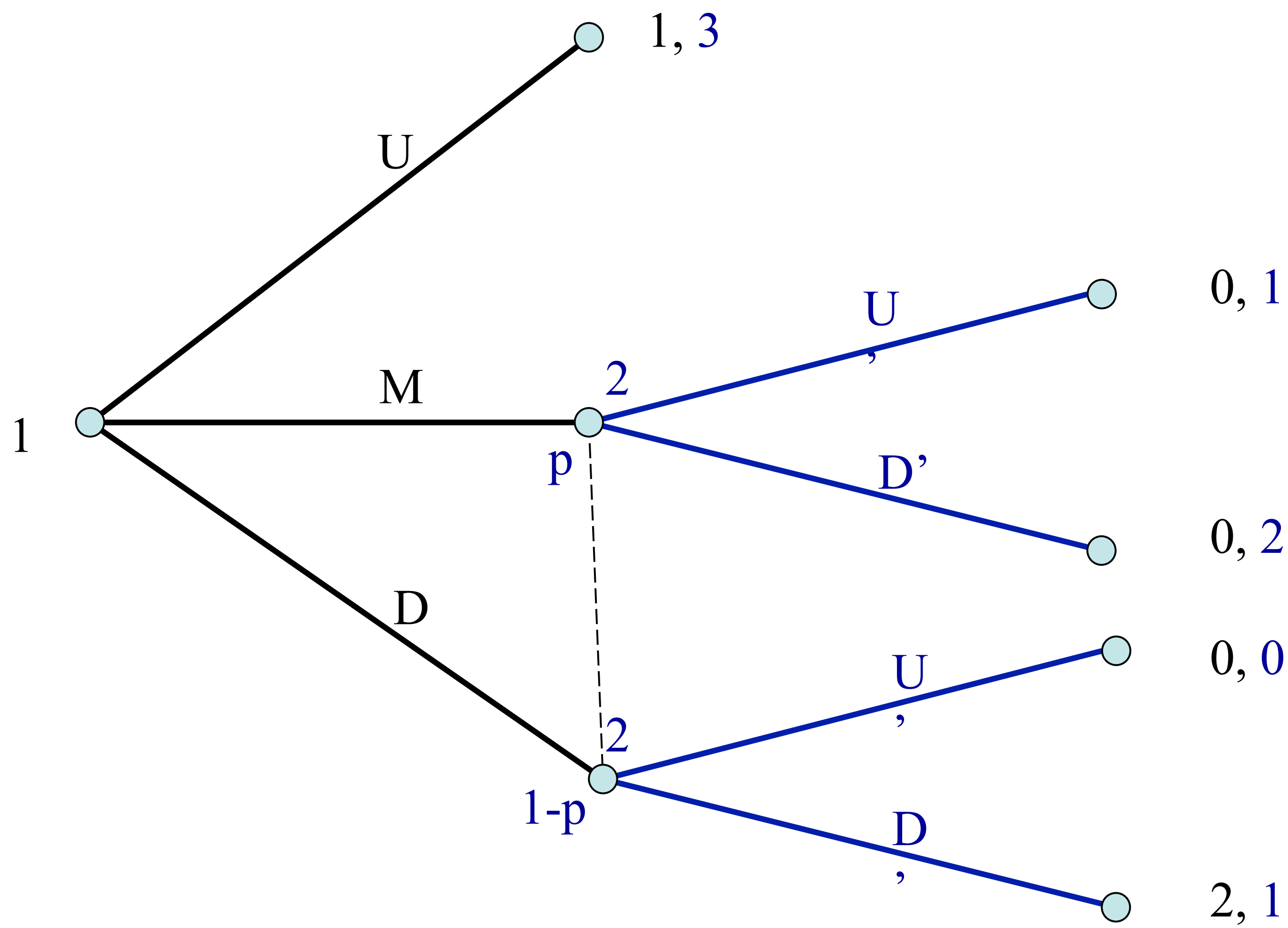
- turn it into a game of complete but imperfect information
- use Nash equilibrium as the solution concept

A Bayes-Nash equilibrium of the game in pure strategies is a triple (s_1,s_{2M},s_{2MM}) in which each player – and each player type - plays a best response, as follows:

(1) s_{2i} maximizes a type i player 2's payoffs when s_1 is 1's strategy;

(2) s_1 maximizes player 1's payoffs when a type i player 2 is playing s_{2i} and the probabilities of types 1 and 2 are respectively p and 1-p

Perfect Bayesian Equilibrium: motivating example



Perfect Bayesian Equilibrium: motivating example

1 \ 2	U'	D'
U	1,3	1,3
M	0,1	0,2
D	0,0	2,1

Perfect Bayesian Equilibrium: motivating example

Subgames: this game has no proper subgames. So any NE is an SPNE. In particular, (U, U’) and (D, D’) are SPNE.

Note: (U, U’) clearly depends on a non-credible threat; if player 2 gets to move, playing D’ dominates U’, so player 1 should not be induced to play U by player 2’s threat to play U’.

Perfect Bayesian Equilibrium: Definition

Strenghtening the Equilibrium Concept

PBE Requirement 1. At each information set, the player who moves must have a belief about which node in the information set has been reached.

PBE Requirement 2. Given their beliefs, the players' strategies must be sequentially rational.

That is, the players' actions must be optimal given the player's belief at that information set and the other players' subsequent strategies.

Perfect Bayesian Equilibrium: motivating example

Player 2 must have a belief about whether player 1 has played M or D. This belief is represented by p and 1 − p attached to the relevant nodes.

The expected payoff from playing
- U' is $p.1 + (1 - p).0 = p$
- D' is $p.2 + (1 - p) \cdot 1 = 1 + p$

Player 2 always chooses D' – we can eliminate (U, U').

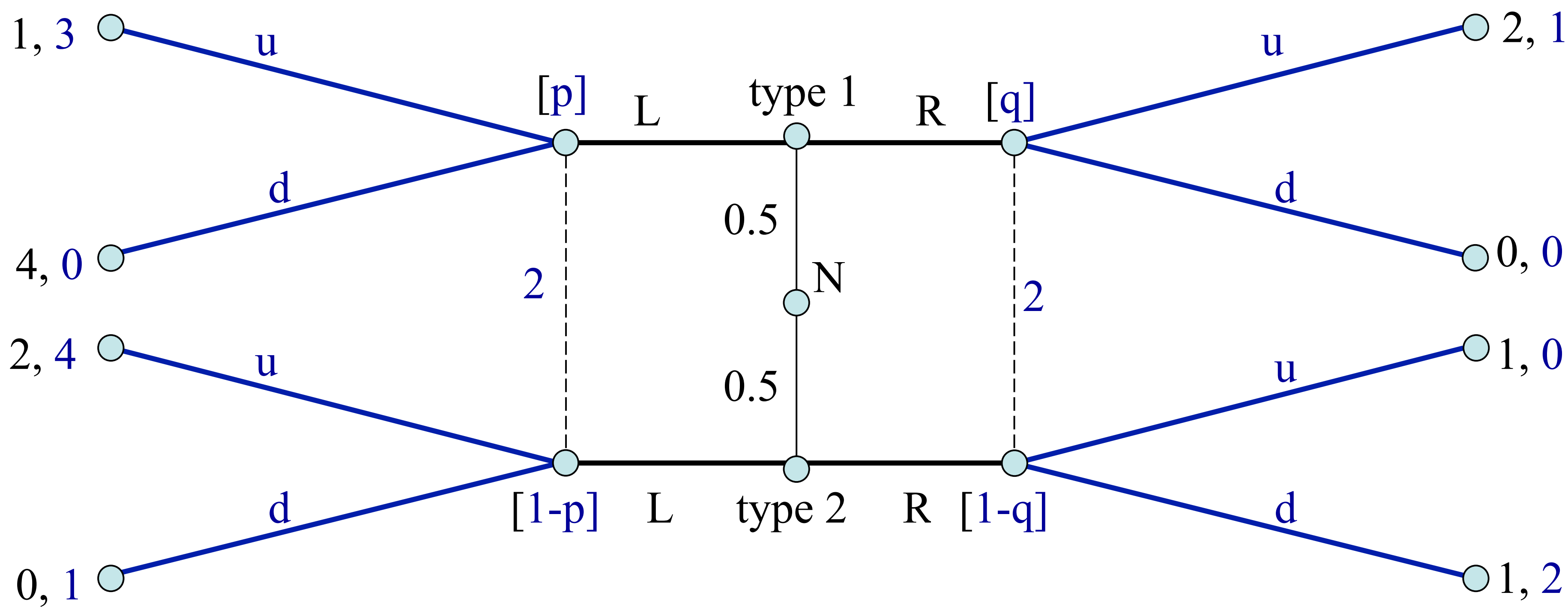
Perfect Bayesian Equilibrium: definition

Def: For a given equilibrium, an information set is on the equilibrium path if it will be reached with positive probability if the game is played according to the equilibrium strategies; it is off the equilibrium path if it is certain not to be reached.

PBE Requirement 3. At information sets on the equilibrium paths, beliefs are determined according to Bayes' rule and the players' equilibrium strategies.

Ex: In the SPNE (D, D'), player 2's belief must be p = 0:

Perfect Bayesian Equilibrium: example



Perfect Bayesian Equilibrium: example

There are four possible pure-strategy perfect Bayesian equilibria in this game:

(1) pooling on L (i.e, both t_1 and t_2 play L)

(2) pooling on R (i.e, both t_1 and t_2 play R)

(3) separation with t_1 playing L and t_2 playing R

(4) separation with t_1 playing R and t_2 playing L

Perfect Bayesian Equilibrium: example

Pooling on L

Suppose 1's strategy is (L, L).

Then 2's information set after L is on the equilibrium path, and 2's belief (p,1-p) is determined by Bayes' rule and 1's strategy. Clearly, we must have $p = 0.5$ (due to pooling).

Given this belief, 2's best response is to play u, so that types t_1 and t_2 earn 1 and 2, respectively.

Is 1 willing to choose (L, L)? If 2's response to R is u, the payoff of t_1 is $2 > 1$ (deviation incentive). If it is d, the payoffs for t_1 and t_2 are $0 < 1$ and $1 < 2$.

Under what conditions is d the optimal choice of 2?

2's expected payoff from d is larger than from u iff $q \cdot 0 + (1 - q) \cdot 2 \geq q \cdot 1 + (1 - q) \cdot 0 \rightarrow q \leq 2/3$

So [(L, L), (u, d), $p = 0.5$, q] is a pooling PBE for $q \leq 2/3$.

Perfect Bayesian Equilibrium: example

Pooling on R

Suppose 1 adopts strategy (R, R).

Clearly, we must have q = 0.5 (due to pooling). Given this belief, 2's best response is to play d, so that types t_1 and t_2 earn 0 and 1, respectively. But t_1 can earn 1 by playing L, since 2's best response to L is u for any value of p.

So there cannot be an equilibrium where 1 plays (R, R).

Perfect Bayesian Equilibrium: example

Separating, with t_1 playing L

Suppose 1 adopts strategy (L, R).

Then both of 2's information sets are on the equilibrium path, so both beliefs are determined using Bayes' rule and the eq. strategy: $p = 1$, $q = 0$.

2's best responses to these beliefs are u and d, respectively, and both types earn 1.

Is (L, R) optimal given 2's strategy (u, d)? No: if type t_2 deviates by playing L rather than R, 2 responds with u, earning t_2 a payoff of $2 > 1$ (deviation incentive).

So there cannot be an equilibrium where 1 plays (L, R).

Perfect Bayesian Equilibrium: example

Separating, with t_1 playing R

Suppose 1 adopts strategy (R, L).

2's beliefs are reversed: $p = 0$, $q = 1$. 2's best response is (u, u) and both types earn payoffs of 2.

If t_1 were to deviate by playing L rather than R, 2 would react with u, and t_1's payoff would be $1 < 2$. So there is no incentive to deviate for t_1.

If t_2 were to deviate by playing R rather than L, 2 would react with u, and t_2's payoff would be $1 < 2$. So there is no incentive to deviate for t_2.

So there is a separating PBE [(R, L), (u, u), $p = 0$, $q = 1$].